

3-quasi-Sasakian manifolds

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Abstract

In the present paper we carry on a systematic study of 3-quasi-Sasakian manifolds. In particular we prove that the three Reeb vector fields generate an involutive distribution determining a canonical totally geodesic and Riemannian foliation. Locally, the leaves of this foliation turn out to be Lie groups: either the orthogonal group or an abelian one. We show that 3-quasi-Sasakian manifolds have a well-defined rank, obtaining a rank-based classification. Furthermore, we prove a splitting theorem for these manifolds assuming the integrability of one of the almost product structures. Finally, we show that the vertical distribution is a minimum of the corrected energy.

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1 Introduction

Quasi-Sasakian manifolds were introduced and deeply studied by D. E. Blair in [1] in the attempt to unify Sasakian and cosymplectic geometry. He defined a quasi-Sasakian structure on a $(2n+1)$ -dimensional manifold M as a normal almost contact metric structure (ϕ, ξ, η, g) whose fundamental 2-form Φ is closed. His work was aimed especially to explore the geometric meaning of the rank of the 1-form η . For instance, in a cosymplectic manifold η has rank 1 and in a Sasakian manifold it has maximal rank $2n+1$. Blair proved that there are no quasi-Sasakian manifolds of even rank. So, let $2p+1$ be the rank of a quasi-Sasakian manifold; in this case, if the determined almost product structure is integrable, the space is locally the product of a Sasakian manifold where $\eta \wedge (d\eta)^p \neq 0$ and a Kähler manifold whose fundamental 2-form is $\Phi - d\eta$ properly restricted.

Next, some other general results were found by S. Tanno [23], S. Kanemaki [13], Z. Olszak [19] and other authors. More recently quasi-Sasakian manifolds appeared also in other contexts such as CR-geometry and mathematical physics (see e.g.

[5, 11, 12, 20]). A study of quasi-Sasakian structures using the techniques of G -structures by V. F. Kirichenko and A. R. Rustanov should also be mentioned [16].

In the same years when the theory of quasi-Sasakian structures was developed, the Japanese school including geometers such as Y. Y. Kuo, K. Yano, M. Konishi, S. Ishihara et al. introduced the notion of almost 3-contact manifold as a triple of almost contact structures on a $(4n + 3)$ -dimensional manifold M satisfying the proper generalization of the quaternionic identities. From then on, several authors in the last 30–40 years have dealt with almost 3-contact geometry, in particular in the setting of 3-Sasakian manifolds, due to the increasing awareness of their importance in mathematics and physics, together with the closely linked hyper-Kählerian and quaternion-Kählerian manifolds. We refer the reader to the remarkable survey [6] and references therein.

When each of the three structures of an almost 3-contact metric manifold is quasi-Sasakian, we say that the manifold in question is endowed with a 3-quasi-Sasakian structure. Although the notion of 3-quasi-Sasakian structure is well known in literature, it seems that a systematic study of these manifolds has not been conducted so far. This is what we plan to do in the present paper.

The first step in our work is to prove that the distribution spanned by the three characteristic vector fields ξ_1, ξ_2, ξ_3 of a 3-quasi-Sasakian manifold is involutive and defines a 3-dimensional Riemannian and totally geodesic foliation \mathcal{V} of M . This property, in general, does not hold for an almost 3-contact metric manifold as we show by a pair of simple examples. Taking into account the geometry of the foliation \mathcal{V} we show that 3-quasi-Sasakian manifolds divide into two classes: those manifolds for which the foliation \mathcal{V} has the local structure of an abelian Lie group, and those for which \mathcal{V} has the local structure of the Lie group $SO(3)$ (or $SU(2)$).

For a 3-quasi-Sasakian manifold one can consider the ranks of the three structures $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$. We will prove that these ranks coincide, allowing us to classify 3-quasi-Sasakian manifolds according to their well-defined rank, which is of the form $4p + 1$ in the abelian case and $4p + 3$ in the non-abelian one. Note that 3-cosymplectic manifolds (rank 1) and 3-Sasakian manifolds (rank $4n + 3 = \dim(M)$) are two representatives of each of the above classes. Nevertheless we show examples of 3-quasi-Sasakian manifolds which are neither 3-cosymplectic nor 3-Sasakian. In this sense we can say that the notion of 3-quasi-Sasakian manifold is a natural generalization of that of 3-Sasakian and that of 3-cosymplectic manifold.

Furthermore, we prove a splitting theorem for any 3-quasi-Sasakian manifold M assuming the integrability of one of the almost product structures. We prove that if M belongs to the class of 3-quasi-Sasakian manifolds with $[\xi_\alpha, \xi_\beta] = 2\xi_\gamma$, then M is locally the product of a 3-Sasakian and a hyper-Kählerian manifold, whereas if M belongs to the class of 3-quasi-Sasakian manifolds with $[\xi_\alpha, \xi_\beta] = 0$, then M is 3-cosymplectic. Finally, we find an application of the integrability of the vertical

distribution showing that \mathcal{V} is a minimum of the corrected energy in the sense of Chacón-Naveira-Weston [8, 9] and Blair-Turgut Vanli [3, 4].

2 Preliminaries

By an *almost contact manifold* we mean an odd-dimensional manifold M which carries a field ϕ of endomorphisms of the tangent spaces, a vector field ξ , called *characteristic* or *Reeb vector field*, and a 1-form η satisfying $\phi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$, where $I: TM \rightarrow TM$ is the identity mapping. From the definition it follows also that $\phi\xi = 0$, $\eta \circ \phi = 0$ and that the $(1, 1)$ -tensor field ϕ has constant rank $2n$ where $2n + 1$ is the dimension of M (cf. [2]). Given an almost contact manifold (M, ϕ, ξ, η) one can define an almost complex structure J on the product $M \times \mathbb{R}$ by setting $J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X) \frac{d}{dt})$ for any $X \in \Gamma(TM)$ and $f \in C^\infty(M \times \mathbb{R})$. Then the almost contact manifold is said to be *normal* if the almost complex structure J is integrable. The computation of the Nijenhuis tensor of J gives rise to the four tensors defined by

$$\begin{aligned} N^{(1)}(X, Y) &= [\phi, \phi](X, Y) + 2d\eta(X, Y)\xi, \\ N^{(2)}(X, Y) &= (\mathcal{L}_{\phi X}\eta)(Y) - (\mathcal{L}_{\phi Y}\eta)(X), \\ N^{(3)}(X) &= (\mathcal{L}_\xi\phi)X, \\ N^{(4)}(X) &= (\mathcal{L}_\xi\eta)(X), \end{aligned}$$

where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ and \mathcal{L}_X denotes the Lie derivative with respect to the vector field X . One finds that the structure (ϕ, ξ, η) is normal if and only if $N^{(1)}$ vanishes identically; in particular, $N^{(1)} = 0$ implies the vanishing of $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$ (cf. [22]).

Any almost contact manifold (M, ϕ, ξ, η) admits a *compatible* Riemannian metric g such that $g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$ for all $X, Y \in \Gamma(TM)$. The manifold M is then said to be an *almost contact metric manifold* with structure (ϕ, ξ, η, g) . The 2-form Φ on M defined by $\Phi(X, Y) = g(X, \phi Y)$ is called the *fundamental 2-form* of the almost contact metric manifold (M, ϕ, ξ, η, g) .

Almost contact metric manifolds such that both η and Φ are closed are called *almost cosymplectic manifolds* and those such that $d\eta = \Phi$ are called *contact metric manifolds*. Finally, a *cosymplectic manifold* is a normal almost cosymplectic manifold while a *Sasakian manifold* is a normal contact metric manifold.

The notion of quasi-Sasakian structure, introduced by D. E. Blair in [1], unifies those of Sasakian and cosymplectic structures. A *quasi-Sasakian manifold* is defined as a normal almost contact metric manifold whose fundamental 2-form is closed. A quasi Sasakian manifold M is said to be of rank $2p$ (for some $p \leq n$) if $(d\eta)^p \neq 0$

and $\eta \wedge (d\eta)^p = 0$ on M , and to be of rank $2p + 1$ if $\eta \wedge (d\eta)^p \neq 0$ and $(d\eta)^{p+1} = 0$ on M (cf. [1, 23]). Blair proved that there are no quasi-Sasakian manifolds of even rank. If the rank of M is $2p + 1$, then the module $\Gamma(TM)$ of vector fields over M splits into two submodules as follows: $\Gamma(TM) = \mathcal{E}^{2p+1} \oplus \mathcal{E}^{2q}$, $p + q = n$, where

$$\mathcal{E}^{2q} = \{X \in \Gamma(TM) \mid i_X d\eta = 0 \text{ and } i_X \eta = 0\}$$

and $\mathcal{E}^{2p+1} = \mathcal{E}^{2p} \oplus \langle \xi \rangle$, \mathcal{E}^{2p} being the orthogonal complement of $\mathcal{E}^{2q} \oplus \langle \xi \rangle$ in $\Gamma(TM)$. These modules satisfy $\phi \mathcal{E}^{2p} = \mathcal{E}^{2p}$ and $\phi \mathcal{E}^{2q} = \mathcal{E}^{2q}$ (cf. [23]).

We will now mention some properties of quasi-Sasakian manifolds which will be useful in the sequel.

Lemma 2.1 ([13]) *A necessary and sufficient condition for an almost contact metric manifold (M, ϕ, ξ, η, g) to be quasi-Sasakian is that there exists a symmetric linear transformation field A on M , commuting with ϕ , such that*

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad (1)$$

for any vector fields X and Y on M , where ∇ denotes the Levi-Civita connection.

Replacing Y with ξ in (1) and applying ϕ we easily get

$$\phi A = \nabla \xi, \quad (2)$$

from which it follows that $\nabla_\xi \xi = \phi A \xi = A \phi \xi = 0$, so the integral curves of ξ are geodesics. Moreover, the vanishing of $N^{(2)}$ implies

$$d\eta(\phi X, Y) = d\eta(\phi Y, X) \quad (3)$$

for all $X, Y \in \Gamma(TM)$. In quasi-Sasakian manifolds the Reeb vector field has the following properties.

Lemma 2.2 ([1],[19]) *Let (M, ϕ, ξ, η, g) be a quasi-Sasakian manifold. Then*

- (i) *the Reeb vector field ξ is Killing;*
- (ii) *the Ricci curvature in the direction of ξ is given by*

$$\text{Ric}(\xi) = \|\nabla \xi\|^2. \quad (4)$$

We now come to the main topic of our paper, i.e. 3-quasi-Sasakian geometry, which is framed into the more general setting of almost 3-contact geometry. An *almost 3-contact manifold* is a $(4n + 3)$ -dimensional smooth manifold M endowed

with three almost contact structures (ϕ_1, ξ_1, η_1) , (ϕ_2, ξ_2, η_2) , (ϕ_3, ξ_3, η_3) satisfying the following relations, for any even permutation (α, β, γ) of $\{1, 2, 3\}$,

$$\begin{aligned}\phi_\gamma &= \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta, \\ \xi_\gamma &= \phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha.\end{aligned}\tag{5}$$

This notion was introduced by Y. Y. Kuo [17] and, independently, by C. Udriste [24]. Kuo proved that given an almost contact 3-structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$, there exists a Riemannian metric g compatible with each of them and hence we can speak of *almost contact metric 3-structures*. It is well known that in any almost 3-contact metric manifold the Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal with respect to the compatible metric g and that the structural group of the tangent bundle is reducible to $Sp(n) \times I_3$. Moreover, by putting $\mathcal{H} = \bigcap_{\alpha=1}^3 \ker(\eta_\alpha)$ one obtains a $4n$ -dimensional distribution on M and the tangent bundle splits as the orthogonal sum $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle$. We will call any vector belonging to the distribution \mathcal{H} *horizontal* and any vector belonging to the distribution \mathcal{V} *vertical*. An almost 3-contact manifold M is said to be *hyper-normal* if each almost contact structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha)$ is normal. It should be remarked that, as it was proved in [26], if two of the almost contact structures are normal, then so is the third.

Definition 2.3 *A 3-quasi-Sasakian manifold is a hyper-normal almost 3-contact metric manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ such that each fundamental 2-form Φ_α is closed.*

The class of 3-quasi-Sasakian manifolds includes as special cases the well-known 3-Sasakian and 3-cosymplectic manifolds.

All manifolds considered here are assumed to be smooth i.e. of the class \mathcal{C}^∞ , and connected; we denote by $\Gamma(\cdot)$ the set of all sections of a corresponding bundle. We use the convention that $2u \wedge v = u \otimes v - v \otimes u$.

3 The canonical foliation of a 3-quasi-Sasakian manifold

In this section we deal with the integrability of the distribution \mathcal{V} generated by the Reeb vector fields of an almost 3-contact manifold. We exhibit two examples of non-hyper-normal almost 3-contact structures. In the first one \mathcal{V} is not involutive, while in the second example we have an integrable distribution.

Example 3.1 Let \mathfrak{g} be the 7-dimensional Lie algebra with basis $\{X_1, X_2, X_3, X_4, \xi_1, \xi_2, \xi_3\}$ and with the Lie brackets defined by $[X_i, X_j] = [X_i, \xi_\alpha] = 0$ and $[\xi_1, \xi_2] = [\xi_2, \xi_3] = [\xi_3, \xi_1] = X_1$, for all $i, j \in \{1, 2, 3, 4\}$ and $\alpha \in \{1, 2, 3\}$. Let G be a Lie group whose Lie algebra is \mathfrak{g} . We define a left-invariant almost 3-contact structure

$(\phi_\alpha, \xi_\alpha, \eta_\alpha)$ on G putting $\phi_\alpha \xi_\beta = \epsilon_{\alpha\beta\gamma} \xi_\gamma$, where $\epsilon_{\alpha\beta\gamma}$ is the totally antisymmetric symbol, and

$$\begin{aligned} \phi_1 X_1 &= X_2, \quad \phi_1 X_2 = -X_1, \quad \phi_1 X_3 = X_4, \quad \phi_1 X_4 = -X_3, \\ \phi_2 X_1 &= X_3, \quad \phi_2 X_2 = -X_4, \quad \phi_2 X_3 = -X_1, \quad \phi_2 X_4 = X_2, \\ \phi_3 X_1 &= X_4, \quad \phi_3 X_2 = X_3, \quad \phi_3 X_3 = -X_2, \quad \phi_3 X_4 = -X_1, \end{aligned} \quad (6)$$

and setting $\eta_\alpha(X_i) = 0$, $\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}$, for all $\alpha, \beta \in \{1, 2, 3\}$ and $i \in \{1, 2, 3, 4\}$. We note that this structure is not hyper-normal. Indeed, a straightforward computation yields $N_1^{(1)}(\xi_1, \xi_2) = -X_1 + X_2 \neq 0$ since X_1 and X_2 are linearly independent.

Example 3.2 Using the above notations, consider the 7-dimensional Lie algebra \mathfrak{g}' with brackets defined by $[X_i, X_j] = [\xi_\alpha, \xi_\beta] = 0$ and $[\xi_\alpha, X_i] = \xi_\alpha$, for each $\alpha \in \{1, 2, 3\}$ and $i, j \in \{1, 2, 3, 4\}$. Let G' be a Lie group whose Lie algebra is \mathfrak{g}' and let η_α and ϕ_α be the same tensors as above. Then $(G', \phi_\alpha, \xi_\alpha, \eta_\alpha)$ is an almost 3-contact manifold which is not hyper-normal because $N_\alpha^{(3)}(X_i) = [\xi_\alpha, \phi_\alpha X_i] - \phi_\alpha[\xi_\alpha, X_i] = \pm \xi_\alpha$, from which it follows that $N_\alpha^{(1)} \neq 0$.

Therefore it makes sense to ask whether for a certain class of almost 3-contact manifolds the distribution \mathcal{V} is integrable. We will prove that this happens for a 3-quasi-Sasakian manifold. To this end we need the following Lemma that is a straightforward generalization of Lemma 2.1.

Lemma 3.3 *An almost 3-contact metric manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is a 3-quasi-Sasakian manifold if and only if it carries three symmetric tensor fields A_1, A_2, A_3 such that, for each $\alpha \in \{1, 2, 3\}$,*

$$(\nabla_X \phi_\alpha)Y = \eta_\alpha(Y)A_\alpha X - g(A_\alpha X, Y)\xi_\alpha, \quad \phi_\alpha A_\alpha = A_\alpha \phi_\alpha,$$

for all $X, Y \in \Gamma(TM)$.

Theorem 3.4 *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold. Then the 3-dimensional distribution \mathcal{V} generated by ξ_1, ξ_2, ξ_3 is integrable. Moreover, \mathcal{V} defines a totally geodesic and Riemannian foliation of M .*

Proof. First note that for any $\alpha \in \{1, 2, 3\}$ we have $\nabla_{\xi_\alpha} \xi_\alpha = 0$, each structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ being quasi-Sasakian. Now, applying the first formula in Lemma 3.3 to $X = Y = \xi_\beta$ we obtain, for any $\alpha \neq \beta$,

$$\nabla_{\xi_\beta}(\phi_\alpha \xi_\beta) - \phi_\alpha(\nabla_{\xi_\beta} \xi_\beta) = \eta_\alpha(\xi_\beta)A_\alpha \xi_\beta - g(A_\alpha \xi_\beta, \xi_\beta)\xi_\alpha$$

and hence

$$\nabla_{\xi_\beta} \xi_\gamma = -g(A_\alpha \xi_\beta, \xi_\beta) \xi_\alpha. \quad (7)$$

Thus, the distribution \mathcal{V} is integrable with totally geodesic leaves. Finally, \mathcal{V} is a Riemannian foliation since the Reeb vector fields are Killing. ■

Remark 3.5 Theorem 3.4 can be regarded as an analogue of a result of Kuo and Tachibana [18] stating that the distribution spanned by the three Reeb vector fields of a hyper-normal 3-contact manifold is integrable with totally geodesic leaves.

Corollary 3.6 *In any 3-quasi-Sasakian manifold the operators A_α preserve the vertical and horizontal distributions.*

Proof. Applying the first formula in Lemma 3.3 to $X = Y = \xi_\alpha$ one easily gets $A_\alpha \xi_\alpha = g(A_\alpha \xi_\alpha, \xi_\alpha) \xi_\alpha$. For $\alpha \neq \beta$, applying the formula $A_\alpha \phi_\alpha = \nabla \xi_\alpha$ and (7) we obtain

$$A_\alpha \xi_\beta = -A_\alpha \phi_\alpha \xi_\gamma = -\nabla_{\xi_\gamma} \xi_\alpha = g(A_\beta \xi_\gamma, \xi_\gamma) \xi_\beta.$$

Thus, the distribution \mathcal{V} is preserved by A_α . Moreover, since the operators A_α are symmetric they also preserve the horizontal distribution \mathcal{H} . ■

Corollary 3.7 *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold. Then, for all $X \in \Gamma(\mathcal{H})$ and for all $\alpha, \beta \in \{1, 2, 3\}$, $d\eta_\alpha(X, \xi_\beta) = 0$.*

Proof. For $\alpha = \beta$ the result is proved in [1]. Next, let $\alpha \neq \beta$. Using the integrability of the distribution \mathcal{V} and the fact that ξ_β is Killing we have $2d\eta_\alpha(X, \xi_\beta) = -\eta_\alpha([X, \xi_\beta]) = -g([X, \xi_\beta], \xi_\alpha) = \xi_\beta(g(X, \xi_\alpha)) - g(X, [\xi_\beta, \xi_\alpha]) = 0$. ■

Corollary 3.8 *In any 3-quasi-Sasakian manifold each Reeb vector field ξ_α is an infinitesimal automorphism with respect to the horizontal distribution \mathcal{H} .*

Proof. From Corollary 3.7 it follows that for any $X \in \Gamma(\mathcal{H})$ and any $\alpha, \beta \in \{1, 2, 3\}$, $0 = d\eta_\alpha(X, \xi_\beta) = -\frac{1}{2}\eta_\alpha([X, \xi_\beta])$ and hence $[X, \xi_\beta] \in \Gamma(\mathcal{H})$. ■

4 Lie group structures on the leaves of \mathcal{V}

In this section we prove that the canonical foliation \mathcal{V} of a 3-quasi-Sasakian manifold has locally either the structure of the orthogonal group or that of an abelian group. Accordingly, 3-quasi-Sasakian manifolds split into two main classes.

Lemma 4.1 *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a hyper-normal almost 3-contact metric manifold. Then, for any even permutation (α, β, γ) of $\{1, 2, 3\}$ we have*

$$[\xi_\alpha, \xi_\beta] = [\xi_\alpha, \xi_\beta]_{\mathcal{H}} + f\xi_\gamma,$$

where f is the function given by $f = \eta_\gamma([\xi_\alpha, \xi_\beta]) = \eta_\alpha([\xi_\beta, \xi_\gamma]) = \eta_\beta([\xi_\gamma, \xi_\alpha])$.

Proof. Since M is hyper-normal we have that $\mathcal{L}_{\xi_\alpha}\phi_\alpha = 0$. The computation of $(\mathcal{L}_{\xi_\alpha}\phi_\alpha)\xi_\gamma = 0$ yields $[\xi_\alpha, \xi_\beta] = -\phi_\alpha[\xi_\alpha, \xi_\gamma]$, while $(\mathcal{L}_{\xi_\beta}\phi_\beta)\xi_\gamma = 0$ yields $[\xi_\beta, \xi_\alpha] = \phi_\beta[\xi_\beta, \xi_\gamma]$. It follows that $[\xi_\alpha, \xi_\beta] \in \ker(\eta_\alpha) \cap \ker(\eta_\beta)$. Thus

$$[\xi_\alpha, \xi_\beta] = [\xi_\alpha, \xi_\beta]_{\mathcal{H}} + f_\gamma\xi_\gamma, \quad (8)$$

for some functions f_γ . We prove that $f_1 = f_2 = f_3$. Applying ϕ_α to (8) we get

$$\phi_\alpha[\xi_\alpha, \xi_\beta] = \phi_\alpha[\xi_\alpha, \xi_\beta]_{\mathcal{H}} - f_\gamma\xi_\beta.$$

On the other hand, from $(\mathcal{L}_{\xi_\alpha}\phi_\alpha)\xi_\beta = 0$ it follows that

$$\phi_\alpha[\xi_\alpha, \xi_\beta] = [\xi_\alpha, \xi_\gamma] = -[\xi_\gamma, \xi_\alpha]_{\mathcal{H}} - f_\beta\xi_\beta.$$

Hence, $\phi_\alpha[\xi_\alpha, \xi_\beta]_{\mathcal{H}} - f_\gamma\xi_\beta = -[\xi_\gamma, \xi_\alpha]_{\mathcal{H}} - f_\beta\xi_\beta$, from which it follows that $f_\beta = f_\gamma$. ■

Now we prove that for a 3-quasi-Sasakian manifold the function f which appears in Lemma 4.1 is necessarily constant.

Theorem 4.2 *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold. Then, for any even permutation (α, β, γ) of $\{1, 2, 3\}$ and for some $c \in \mathbb{R}$*

$$[\xi_\alpha, \xi_\beta] = c\xi_\gamma.$$

Proof. From Lemma 4.1 and Theorem 3.4 it follows that $[\xi_\alpha, \xi_\beta] = f\xi_\gamma$, where $f \in C^\infty(M)$ is the function given by $f = \eta_\alpha([\xi_\beta, \xi_\gamma]) = -2d\eta_\alpha(\xi_\beta, \xi_\gamma)$. We have thus only to prove that such a function is constant. Indeed, the vanishing of $N^{(4)}$ implies that $\mathcal{L}_{\xi_\alpha}d\eta_\alpha = 0$. Then, for any even permutation (α, β, γ) of $\{1, 2, 3\}$,

$$\begin{aligned} 0 &= (\mathcal{L}_{\xi_\alpha}d\eta_\alpha)(\xi_\beta, \xi_\gamma) \\ &= \xi_\alpha(d\eta_\alpha(\xi_\beta, \xi_\gamma)) - d\eta_\alpha([\xi_\alpha, \xi_\beta], \xi_\gamma) - d\eta_\alpha(\xi_\beta, [\xi_\alpha, \xi_\gamma]) \\ &= -\frac{1}{2}\xi_\alpha(f) - f d\eta_\alpha(\xi_\gamma, \xi_\gamma) + f d\eta_\alpha(\xi_\beta, \xi_\beta) \\ &= -\frac{1}{2}\xi_\alpha(f), \end{aligned}$$

so that f is constant along the leaves of \mathcal{V} . It remains to show that $X(f) = 0$ for all $X \in \Gamma(\mathcal{H})$. In fact, using the formula of the differential of a 2-form, we find

$$\begin{aligned} -\frac{1}{2}X(f) &= X(d\eta_\alpha(\xi_\beta, \xi_\gamma)) \\ &= 3d^2\eta_\alpha(X, \xi_\beta, \xi_\gamma) - \xi_\beta(d\eta_\alpha(\xi_\gamma, X)) - \xi_\gamma(d\eta_\alpha(X, \xi_\beta)) \\ &\quad + d\eta_\alpha([X, \xi_\beta], \xi_\gamma) + d\eta_\alpha([\xi_\beta, \xi_\gamma], X) + d\eta_\alpha([\xi_\gamma, X], \xi_\beta) \\ &= f d\eta_\alpha(\xi_\alpha, X) = 0, \end{aligned}$$

where we have used Corollary 3.7, Corollary 3.8 and the integrability of \mathcal{V} . ■

Using Theorem 4.2 we may therefore divide 3-quasi-Sasakian manifolds in two classes according to the behaviour of the leaves of the canonical foliation \mathcal{V} : those 3-quasi-Sasakian manifolds for which each leaf of \mathcal{V} is locally $SO(3)$ (or $SU(2)$) (which corresponds to take in Theorem 4.2 the constant $c \neq 0$), and those for which each leaf of \mathcal{V} is locally an abelian group (this corresponds to the case $c = 0$).

Note that 3-Sasakian manifolds and 3-cosymplectic manifolds are representatives of each of the above classes. However we emphasize the circumstance that 3-Sasakian and 3-cosymplectic manifolds do not exhaust the above two classes. This can be seen in the construction of the following example.

Example 4.3 Let us denote the canonical global coordinates on \mathbb{R}^{4n+3} by $x_1, \dots, x_n, y_1, \dots, y_n, u_1, \dots, u_n, v_1, \dots, v_n, z_1, z_2, z_3$. We consider the open submanifold M of \mathbb{R}^{4n+3} obtained by removing the points where $\sin(z_2) = 0$ and define three vector fields and three 1-forms on M by

$$\begin{aligned} \xi_1 &= c \frac{\partial}{\partial z_1}, \\ \xi_2 &= c \left(\cos(z_1) \cot(z_2) \frac{\partial}{\partial z_1} + \sin(z_1) \frac{\partial}{\partial z_2} - \frac{\cos(z_1)}{\sin(z_2)} \frac{\partial}{\partial z_3} \right), \\ \xi_3 &= c \left(-\sin(z_1) \cot(z_2) \frac{\partial}{\partial z_1} + \cos(z_1) \frac{\partial}{\partial z_2} + \frac{\sin(z_1)}{\sin(z_2)} \frac{\partial}{\partial z_3} \right) \end{aligned}$$

for some non-zero real number c , and

$$\begin{aligned} \eta_1 &= \frac{1}{c} (dz_1 + \cos(z_2) dz_3), \\ \eta_2 &= \frac{1}{c} (\sin(z_1) dz_2 - \cos(z_1) \sin(z_2) dz_3), \\ \eta_3 &= \frac{1}{c} (\cos(z_1) dz_2 + \sin(z_1) \sin(z_2) dz_3). \end{aligned}$$

A simple computation shows that $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ for any cyclic permutation, and $\eta_\alpha(\xi_\beta) = \delta_{\alpha\beta}$ for any $\alpha, \beta \in \{1, 2, 3\}$. Now, in order to define three tensor fields ϕ_α and a Riemannian metric g on M , we set, for each $i \in \{1, \dots, n\}$, $X_i = \frac{\partial}{\partial x_i}$, $Y_i = \frac{\partial}{\partial y_i}$, $U_i = \frac{\partial}{\partial u_i}$, $V_i = \frac{\partial}{\partial v_i}$. Let g be the Riemannian metric with respect to which $\{X_1, \dots, X_n, Y_1, \dots, Y_n, U_1, \dots, U_n, V_1, \dots, V_n, \xi_1, \xi_2, \xi_3\}$ is a (global) orthonormal frame, and ϕ_1, ϕ_2, ϕ_3 be the tensor fields defined by putting $\phi_\alpha \xi_\beta = \epsilon_{\alpha\beta\gamma} \xi_\gamma$ and, for all $i \in \{1, \dots, n\}$,

$$\begin{aligned} \phi_1 X_i &= Y_i, \quad \phi_1 Y_i = -X_i, \quad \phi_1 U_i = V_i, \quad \phi_1 V_i = -U_i, \\ \phi_2 X_i &= U_i, \quad \phi_2 Y_i = -V_i, \quad \phi_2 U_i = -X_i, \quad \phi_2 V_i = Y_i, \\ \phi_3 X_i &= V_i, \quad \phi_3 Y_i = U_i, \quad \phi_3 U_i = -Y_i, \quad \phi_3 V_i = -X_i. \end{aligned} \tag{9}$$

Note that we have also $\eta_\alpha(X_i) = \eta_\alpha(Y_i) = \eta_\alpha(U_i) = \eta_\alpha(V_i) = 0$. By a lengthy computation that we omit one can verify that $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is an almost contact metric 3-structure. Observe that the structure is not 3-cosymplectic since the Reeb vector fields do not commute. Nevertheless it is neither 3-Sasakian, since it admits a Darboux-like coordinate system (cf. [7]). Some simple computations show that $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is hyper-normal. For instance,

$$\begin{aligned} N_1^{(1)}(\xi_2, \xi_3) &= \phi_1^2[\xi_2, \xi_3] + [\phi_1 \xi_2, \phi_1 \xi_3] - \phi_1[\xi_2, \phi_1 \xi_3] - \phi_1[\phi_1 \xi_2, \xi_3] - \eta_1([\xi_2, \xi_3])\xi_1 \\ &= c\phi_1^2 \xi_1 - [\xi_3, \xi_2] + \phi_1[\xi_2, \xi_2] - \phi_1[\xi_3, \xi_3] - c\eta_1(\xi_1)\xi_1 = 0. \end{aligned}$$

Furthermore, the fundamental 2-forms Φ_α are closed. Indeed, for instance we have

$$\begin{aligned} 3d\Phi_1(X_i, X_j, X_k) &= X_i(\Phi_1(X_j, X_k)) - X_j(\Phi_1(X_i, X_k)) + X_k(\Phi_1(X_i, X_j)) \\ &= X_i(g(X_j, \phi_1 X_k)) - X_j(g(X_i, \phi_1 X_k)) + X_k(g(X_i, \phi_1 X_j)) = 0, \end{aligned}$$

since the scalar products in the above expression are constant functions. Thus, $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is a 3-quasi-Sasakian manifold. Furthermore, one can show that M is η -Einstein, its Ricci tensor being given by $\text{Ric} = \frac{c^2}{2}(\eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \eta_3 \otimes \eta_3)$. Thus, differently from 3-Sasakian and 3-cosymplectic geometry, there are 3-quasi-Sasakian manifolds which are not Einstein.

5 The rank of a 3-quasi-Sasakian manifold

For a 3-quasi-Sasakian manifold one can consider the ranks of the three structures $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$. In this section we will prove that these ranks coincide allowing us to classify 3-quasi-Sasakian manifolds. First we need some technical results.

Lemma 5.1 *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be an almost 3-contact metric manifold. Then we have*

$$\eta_\alpha = \frac{1}{2} i_{\xi_\beta} \Phi_\gamma, \quad (10)$$

for an even permutation (α, β, γ) of $\{1, 2, 3\}$. Furthermore, if the fundamental 2-forms Φ_α are closed, we have

$$d\eta_\alpha = \frac{1}{2} \mathcal{L}_{\xi_\beta} \Phi_\gamma. \quad (11)$$

Proof. For any $X \in \Gamma(TM)$ we have $\eta_\alpha(X) = g(X, \xi_\alpha) = -g(X, \phi_\gamma \xi_\beta) = -\Phi_\gamma(X, \xi_\beta) = \frac{1}{2} i_{\xi_\beta} \Phi_\gamma(X)$. Assuming $d\Phi_\alpha = 0$ and applying the Cartan formula we get $\mathcal{L}_{\xi_\beta} \Phi_\gamma = di_{\xi_\beta} \Phi_\gamma + i_{\xi_\beta} d\Phi_\gamma = di_{\xi_\beta} \Phi_\gamma = 2d\eta_\alpha$. ■

Lemma 5.2 *In any 3-quasi-Sasakian manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ we have, for any even permutation (α, β, γ) of $\{1, 2, 3\}$ and for all $X, Y \in \Gamma(\mathcal{H})$,*

$$d\eta_\alpha(\phi_\beta X, Y) = d\eta_\gamma(X, Y).$$

Proof. First of all note that $\Phi_\gamma(\phi_\beta X, Y) = g(\phi_\beta X, \phi_\gamma Y) = -g(X, \phi_\beta \phi_\gamma Y) = -g(X, \phi_\alpha Y) = -\Phi_\alpha(X, Y)$. Using this and (11) we get

$$\begin{aligned} 2d\eta_\alpha(\phi_\beta X, Y) &= (\mathcal{L}_{\xi_\beta} \Phi_\gamma)(\phi_\beta X, Y) \\ &= \xi_\beta(\Phi_\gamma(\phi_\beta X, Y)) - \Phi_\gamma([\xi_\beta, \phi_\beta X], Y) - \Phi_\gamma(\phi_\beta X, [\xi_\beta, Y]) \\ &= \xi_\beta(\Phi_\gamma(\phi_\beta X, Y)) - \Phi_\gamma(\phi_\beta [\xi_\beta, X], Y) - \Phi_\gamma(\phi_\beta X, [\xi_\beta, Y]) \\ &= -\xi_\beta(\Phi_\alpha(X, Y)) + \Phi_\alpha([\xi_\beta, X], Y) + \Phi_\alpha(X, [\xi_\beta, Y]) \\ &= -(\mathcal{L}_{\xi_\beta} \Phi_\alpha)(X, Y) \\ &= 2d\eta_\gamma(X, Y), \end{aligned}$$

where we have used the fact that $N_\beta^{(3)} = 0$. ■

The next lemma is an immediate consequence of Lemma 5.2 and Corollary 3.7.

Lemma 5.3 *In any 3-quasi-Sasakian manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ we have, for any $\alpha \neq \beta$ and for all $X, Y \in \Gamma(\mathcal{H})$,*

- (i) $d\eta_\alpha(X, \phi_\alpha Y) = d\eta_\beta(X, \phi_\beta Y),$
- (ii) $d\eta_\alpha(\phi_\beta X, \phi_\beta Y) = -d\eta_\alpha(X, Y).$

Lemma 5.4 *In any 3-quasi-Sasakian manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ we have $i_X d\eta_\alpha = 0$ if and only if $i_X d\eta_\beta = 0$ for any $X \in \Gamma(\mathcal{H})$ and $\alpha, \beta \in \{1, 2, 3\}$.*

Proof. Assume $i_X d\eta_\alpha = 0$. Then, from Lemma 5.2 it follows that $i_X d\eta_\beta(Y) = -2d\eta_\beta(Y, X) = 2d\eta_\alpha(\phi_\gamma Y, X) = -i_X d\eta_\alpha(\phi_\gamma Y) = 0$. ■

Theorem 5.5 *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold of dimension $4n + 3$. Then the 1-forms η_1, η_2 and η_3 have the same rank $4l + 3$ or $4l + 1$, for some $l \leq n$, according to $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ with $c \neq 0$, or $[\xi_\alpha, \xi_\beta] = 0$, respectively.*

Proof. Let us consider the quasi-Sasakian structures $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ and assume that the 1-forms η_α have ranks $2p_\alpha + 1$. Then, according to [23], we have three decompositions

$$\Gamma(TM) = \mathcal{E}^{2p_\alpha+1} \oplus \mathcal{E}^{2q_\alpha},$$

where

$$\mathcal{E}^{2q_\alpha} = \{X \in \Gamma(TM) \mid i_X d\eta_\alpha = 0 \text{ and } i_X \eta_\alpha = 0\}.$$

Let us consider the class of 3-quasi-Sasakian manifolds such that $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ with $c \neq 0$. In this case we have $i_{\xi_\gamma} d\eta_\alpha(\xi_\beta) = -i_{\xi_\beta} d\eta_\alpha(\xi_\gamma) = c$, which implies that $\xi_\beta, \xi_\gamma \notin \mathcal{E}^{2q_\alpha}$ and

$$\mathcal{E}^{2q_\alpha} = \{X \in \Gamma(\mathcal{H}) \mid i_X d\eta_\alpha = 0\}.$$

Hence, by virtue of Lemma 5.4, $\mathcal{E}^{2q_\alpha} = \mathcal{E}^{2q_\beta}$ and $q_\alpha = q_\beta$ for $\alpha, \beta \in \{1, 2, 3\}$. It can be easily seen that $\phi_\alpha \mathcal{E}^{2q} = \mathcal{E}^{2q}$ for $\alpha \in \{1, 2, 3\}$, where we have set $q = q_\alpha$. Furthermore, the restrictions $\phi_\alpha|_{\mathcal{E}^{2q}}$ of the mappings ϕ_α to \mathcal{E}^{2q} define a quaternionic structure in the submodule \mathcal{E}^{2q} which will be denoted by \mathcal{E}^{4m} since its dimension is a multiple of 4. It follows that in this case the rank of M is $4l + 3$, where $l = n - m$.

Let us now suppose $[\xi_\alpha, \xi_\beta] = 0$. In this case $\xi_\beta \in \mathcal{E}^{2q_\alpha}$ for $\beta \neq \alpha$. Indeed, by Corollary 3.7, $i_{\xi_\beta} d\eta_\alpha(X) = 0$ for any $X \in \Gamma(\mathcal{H})$. Moreover, $i_{\xi_\beta} d\eta_\alpha(\xi_\gamma) = 0$ for any $\gamma \in \{1, 2, 3\}$. It follows that

$$\mathcal{E}^{2q_\alpha} = \langle \xi_\beta, \xi_\gamma \rangle \oplus \{X \in \Gamma(\mathcal{H}) \mid i_X d\eta_\alpha = 0\}.$$

This implies again that $q_\alpha = q_\beta$. Moreover, the restrictions of the mappings ϕ_α define as before a quaternionic structure in $\{X \in \Gamma(\mathcal{H}) \mid i_X d\eta_\alpha = 0\}$, whose dimension is then $4m$ for some $m \leq n$. In this case the rank of the manifold will be $4l + 1$, where $l = n - m$. ■

According to Theorem 5.5, we say that a 3-quasi-Sasakian manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ has rank $4l + 3$ or $4l + 1$ if any quasi-Sasakian structure has such rank. We may thus classify 3-quasi-Sasakian manifolds of dimension $4n + 3$, according to their rank. For any $l \in \{0, \dots, n\}$ we have one class of manifolds such that $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ with $c \neq 0$, and one class of manifolds with $[\xi_\alpha, \xi_\beta] = 0$. The total number of classes amounts then to $2n + 2$.

In the following we will use the notation $\mathcal{E}^{4m} := \{X \in \Gamma(\mathcal{H}) \mid i_X d\eta_\alpha = 0\}$, while \mathcal{E}^{4l} will be the orthogonal complement of \mathcal{E}^{4m} in $\Gamma(\mathcal{H})$, $\mathcal{E}^{4l+3} := \mathcal{E}^{4l} \oplus \Gamma(\mathcal{V})$, and $\mathcal{E}^{4m+3} := \mathcal{E}^{4m} \oplus \Gamma(\mathcal{V})$.

We now consider the class of 3-quasi-Sasakian manifolds such that $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ with $c \neq 0$ and let $4l+3$ be the rank. In this case, according to [1], we define for each structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ two $(1,1)$ -tensor fields ψ_α and θ_α by putting

$$\psi_\alpha X = \begin{cases} \phi_\alpha X, & \text{if } X \in \mathcal{E}^{4l+3}; \\ 0, & \text{if } X \in \mathcal{E}^{4m}; \end{cases} \quad \theta_\alpha X = \begin{cases} 0, & \text{if } X \in \mathcal{E}^{4l+3}; \\ \phi_\alpha X, & \text{if } X \in \mathcal{E}^{4m}. \end{cases}$$

Note that, for each $\alpha \in \{1, 2, 3\}$ we have $\phi_\alpha = \psi_\alpha + \theta_\alpha$. Next, we define a new (pseudo-Riemannian, in general) metric \bar{g} on M setting

$$\bar{g}(X, Y) = \begin{cases} -d\eta_\alpha(X, \phi_\alpha Y), & \text{for } X, Y \in \mathcal{E}^{4l}; \\ g(X, Y), & \text{elsewhere.} \end{cases}$$

This definition is well posed by virtue of (3) and of Lemma 5.3. $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, \bar{g})$ is in fact a hyper-normal almost 3-contact metric manifold, in general non-3-quasi-Sasakian. We are now able to prove the following decomposition theorem.

Theorem 5.6 *Let $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold of rank $4l+3$ with $[\xi_\alpha, \xi_\beta] = 2\xi_\gamma$. Assume $[\theta_\alpha, \theta_\alpha] = 0$ for some $\alpha \in \{1, 2, 3\}$ and \bar{g} positive definite on \mathcal{E}^{4l} . Then M^{4n+3} is locally the product of a 3-Sasakian manifold M^{4l+3} and a hyper-Kählerian manifold M^{4m} with $m = n - l$.*

Proof. The almost product structure defined by $-\theta_\alpha^2$ and $-\psi_\alpha^2 + \eta_\alpha \otimes \xi_\alpha$ is integrable (see e.g. [25]), since $[\theta_\alpha, \theta_\alpha] = 0$ implies $[-\theta_\alpha^2, -\theta_\alpha^2] = 0$. Thus, locally M is the product of the manifolds M^{4l+3} and M^{4m} whose localized modules of vector fields are \mathcal{E}^{4l+3} and \mathcal{E}^{4m} , respectively. Since ψ_α and ϕ_α agree on \mathcal{E}^{4l+3} , $(\psi_\alpha, \xi_\alpha, \eta_\alpha)|_{\mathcal{E}^{4l+3}}$ is an almost contact 3-structure. Furthermore, the metric \bar{g} is by definition compatible with the three almost contact structures and $d\eta_\alpha = \bar{\Phi}_\alpha$ on \mathcal{E}^{4l+3} , so $(\psi_\alpha, \xi_\alpha, \eta_\alpha, \bar{g})|_{\mathcal{E}^{4l+3}}$ is a contact metric 3-structure over M^{4l+3} and thus it is 3-Sasakian, by a result of Kashiwada [15]. Since θ_α agrees with ϕ_α on \mathcal{E}^{4m} , the maps $\theta_\alpha|_{\mathcal{E}^{4m}}$ define a quaternionic structure which is compatible with the metric $\bar{g}|_{\mathcal{E}^{4m}}$. Finally, define the 2-forms $\bar{\Theta}_\alpha$ by $\bar{\Theta}_\alpha(X, Y) = \bar{g}(X, \theta_\alpha Y)$ for any $X, Y \in \mathcal{E}^{4m}$. We have $\bar{\Theta}_\alpha = \bar{\Phi}_\alpha|_{\mathcal{E}^{4m}}$ and hence $d\bar{\Theta}_\alpha = 0$. By virtue of Hitchin Lemma [10] the structure defined on M^{4m} turns out to be hyper-Kählerian. ■

We now consider the class of 3-quasi-Sasakian manifolds such that $[\xi_\alpha, \xi_\beta] = 0$ and let $4l+1$ be the rank. In this case we define for each structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ two $(1,1)$ -tensor fields ψ_α and θ_α by putting

$$\psi_\alpha X = \begin{cases} \phi_\alpha X, & \text{if } X \in \mathcal{E}^{4l}; \\ 0, & \text{if } X \in \mathcal{E}^{4m+3}; \end{cases} \quad \theta_\alpha X = \begin{cases} 0, & \text{if } X \in \mathcal{E}^{4l}; \\ \phi_\alpha X, & \text{if } X \in \mathcal{E}^{4m+3}. \end{cases}$$

Note that for each α the maps $-\psi_\alpha^2$ and $-\theta_\alpha^2 + \eta_\alpha \otimes \xi_\alpha$ define an almost product structure which is integrable if and only if $[-\psi_\alpha^2, -\psi_\alpha^2] = 0$ or, equivalently, $[\psi_\alpha, \psi_\alpha] = 0$. Under this assumption the structure turns out to be 3-cosymplectic as it is shown by the following theorem.

Theorem 5.7 *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold of rank $4l + 1$ such that $[\xi_\alpha, \xi_\beta] = 0$ for any $\alpha, \beta \in \{1, 2, 3\}$ and $[\psi_\alpha, \psi_\alpha] = 0$ for some $\alpha \in \{1, 2, 3\}$. Then M is a 3-cosymplectic manifold.*

Proof. From $[\psi_\alpha, \psi_\alpha] = 0$ it follows that the corresponding quasi-Sasakian structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ is cosymplectic, i.e. $d\eta_\alpha = 0$ (see [1], page 339). Then, for any $\beta \neq \alpha$ and any $X \in \Gamma(\mathcal{H})$, $Y \in \Gamma(TM)$ by Lemma 5.4 we obtain $2d\eta_\beta(X, Y) = i_X d\eta_\beta(Y) = 0$. If $X = \xi_\delta$, $Y = \xi_\rho$ for some $\delta, \rho \in \{1, 2, 3\}$, then $2d\eta_\beta(\xi_\delta, \xi_\rho) = \eta_\beta([\xi_\delta, \xi_\rho]) = 0$. Finally, if $X = \xi_\delta$, $Y \in \Gamma(\mathcal{H})$, then $d\eta_\beta(\xi_\delta, Y) = 0$ by Corollary 3.7. We conclude that $d\eta_\beta = 0$ and M is a 3-cosymplectic manifold. ■

6 Corrected energy of 3-quasi Sasakian manifolds

As a single application, we show that the canonical foliation of 3-quasi Sasakian manifolds is a minimum of the *corrected energy*. Recall that the corrected energy $\mathcal{D}(\mathcal{V})$ of a p -dimensional distribution \mathcal{V} on a compact oriented Riemannian manifold (M^m, g) is defined as (cf. [8])

$$\mathcal{D}(\mathcal{V}) = \int_M \left(\sum_{a=1}^m \|\nabla_{e_a} \xi\|^2 + q(q-2)\|\vec{H}_{\mathcal{H}}\|^2 + p^2\|\vec{H}_{\mathcal{V}}\|^2 \right) d\text{vol}, \quad (12)$$

where $\{e_1, \dots, e_m\}$ is a local adapted frame with $e_1, \dots, e_p \in \mathcal{V}_x$ and $e_{p+1}, \dots, e_m \in \mathcal{H}_x = \mathcal{V}_x^\perp$, and $\xi = e_1 \wedge \dots \wedge e_p$ is a p -vector which determines the distribution \mathcal{V} regarded as a section of the Grassmann bundle $G(p, M^m)$ of oriented p -planes in the tangent spaces of M^m . Finally $\vec{H}_{\mathcal{H}}$ and $\vec{H}_{\mathcal{V}}$ are the mean curvatures of the distributions \mathcal{H} and \mathcal{V} given by

$$\vec{H}_{\mathcal{H}} = \sum_{\alpha=1}^p \left(\frac{1}{q} \sum_{i=p+1}^n h_{ii}^\alpha \right) e_\alpha, \quad \vec{H}_{\mathcal{V}} = \sum_{i=p+1}^n \left(\frac{1}{p} \sum_{\alpha=1}^p h_{\alpha\alpha}^i \right) e_i, \quad (13)$$

where the functions $h_{ij}^\alpha = -g(\nabla_{e_i} e_\alpha, e_j)$ define the second fundamental form of the horizontal distribution in the direction e_α , and $h_{\alpha\beta}^i = -g(\nabla_{e_\alpha} e_i, e_\beta)$ define the second fundamental form of the vertical distribution in the direction e_i . The norm of $\nabla \xi$ is given by

$$\sum_a \|\nabla_{e_a} \xi\|^2 = \sum_{i,j,\alpha} (h_{ij}^\alpha)^2 + \sum_{i,\alpha,\beta} (h_{\alpha\beta}^i)^2. \quad (14)$$

It is proved in [8] that if \mathcal{V} is integrable then

$$\mathcal{D}(\mathcal{V}) \geq \int_M \sum_{i,\alpha} c_{i\alpha} d\text{vol}, \quad (15)$$

where $c_{i\alpha}$ is the sectional curvature of the plane spanned by $e_i \in \mathcal{H}$ and $e_\alpha \in \mathcal{V}$.

Theorem 6.1 *The canonical foliation \mathcal{V} defined by the Reeb vector fields ξ_1, ξ_2, ξ_3 of a compact oriented 3-quasi-Sasakian manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ represents a minimum of the corrected energy $\mathcal{D}(\mathcal{V})$.*

Proof. We show that (15) is in fact an equality. Fix an adapted basis of type $\{e_1, \dots, e_{4n}, \xi_1, \xi_2, \xi_3\}$, with $e_i \in \mathcal{H}$. Using (2) we have $h_{ii}^\alpha = -g(\nabla_{e_i} \xi_\alpha, e_i) = -g(\phi_\alpha A_\alpha e_i, e_i)$ which vanishes since A_α is symmetric and commutes with ϕ_α . Moreover, $h_{\alpha\beta}^i = -g(\nabla_{\xi_\alpha} e_i, \xi_\beta) = g(\nabla_{\xi_\alpha} \xi_\beta, e_i) = 0$ since the foliation \mathcal{V} is totally geodesic. It follows that the mean curvatures $\vec{H}_\mathcal{H}$ and $\vec{H}_\mathcal{V}$ vanish. Now we will express the norm of $\nabla \xi$ in terms of the norms of the Reeb vector fields. We know that in a 3-quasi-Sasakian manifold $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$, where c is a real number, possibly zero, and (α, β, γ) is an even permutation of $\{1, 2, 3\}$. Applying the Koszul formula for the Levi-Civita connection, one easily gets $\nabla_{\xi_\alpha} \xi_\beta = \frac{c}{2} \xi_\gamma$. Then, we have

$$\begin{aligned} \|\nabla \xi_\alpha\|^2 &= \sum_i g(\nabla_{e_i} \xi_\alpha, \nabla_{e_i} \xi_\alpha) + g(\nabla_{\xi_\beta} \xi_\alpha, \nabla_{\xi_\beta} \xi_\alpha) + g(\nabla_{\xi_\gamma} \xi_\alpha, \nabla_{\xi_\gamma} \xi_\alpha) \\ &= \sum_i g(\phi_\alpha A_\alpha e_i, \phi_\alpha A_\alpha e_i) + \frac{c^2}{2}. \end{aligned}$$

On the other hand, applying (14) and $h_{\alpha\beta}^i = 0$, we get

$$\sum_a \|\nabla_{e_a} \xi\|^2 = \sum_{i,j,\alpha} g(\nabla_{e_i} \xi_\alpha, e_j)^2 = \sum_{i,j,\alpha} g(\phi_\alpha A_\alpha e_i, e_j)^2 = \sum_{i,\alpha} g(\phi_\alpha A_\alpha e_i, \phi_\alpha A_\alpha e_i),$$

since $A_\alpha \mathcal{H} \subset \mathcal{H}$ by Corollary 3.6. It follows that

$$\sum_a \|\nabla_{e_a} \xi\|^2 = \sum_{\alpha=1}^3 \|\nabla \xi_\alpha\|^2 - \frac{3}{2} c^2.$$

Therefore the expression of the corrected energy of \mathcal{V} , given by (12), reduces to

$$\mathcal{D}(\mathcal{V}) = \int_M \left(\sum_{\alpha=1}^3 \|\nabla \xi_\alpha\|^2 - \frac{3}{2} c^2 \right) d\text{vol}.$$

Now, a direct computation shows that the sectional curvature of the plane spanned by ξ_α and ξ_β is $K(\xi_\alpha, \xi_\beta) = \frac{c^2}{4}$. Hence, by (4) we get

$$\begin{aligned} \sum_{i=1}^{4n} \sum_{\alpha=1}^3 c_{i\alpha} &= \sum_{\alpha=1}^3 \sum_{i=1}^{4n} K(e_i, \xi_\alpha) \\ &= \sum_{\alpha=1}^3 (\text{Ric}(\xi_\alpha) - K(\xi_\alpha, \xi_\beta) - K(\xi_\alpha, \xi_\gamma)) \\ &= \sum_{\alpha=1}^3 \|\nabla \xi_\alpha\|^2 - \frac{3}{2}c^2, \end{aligned}$$

from which the assertion follows. ■

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